Modern approaches to quantum gravity

Solution 12 Fall 2025

1. The bulk to boundary propagator

(a) With the Ansatz $\phi = e^{i\vec{k}\cdot\vec{x}} f_{\vec{k}}(z)$, and using $\Box\phi = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi)$ in Euclidean Poincaré coordinates, it is straight-forward to obtain

$$z^2 f'' - (d-1)zf' - \vec{k}^2 z^2 f = m^2 f \tag{1}$$

In Mathematica, we find the solutions

$$z^{d/2} J_{\Delta-d/2}(-i|\vec{k}|z) \qquad z^{d/2} Y_{\Delta-d/2}(-i|\vec{k}|z) \tag{2}$$

where J_ν and Y_ν are Bessel functions of the first and second kind. The modified Bessel functions I and K are precisely defined in terms of J and Y by including an i factor in the argument. Thus, an equivalent set of solutions is

$$f^{(1)}_{\vec{k}}(z) = z^{d/2} I_{\Delta-d/2}(|\vec{k}|z) \qquad f^{(2)}_{\vec{k}}(z) = z^{d/2} K_{\Delta-d/2}(|\vec{k}|z) \tag{3}$$

(b) As one can find in Wikipedia, $I_\alpha(x) \sim e^x/\sqrt{2\pi x}$ when $x \to \infty$. Thus, $f^{(1)}$ diverges exponentially in the bulk $z \to \infty$.

As you can check in Mathematica (file provided), the normalization of g is such that

$$g_{\vec{k}}(z) \sim z^{d-\Delta} \qquad (z \to 0) \tag{4}$$

This means that

$$\int d^d k \tilde{\phi}'(\vec{k}) e^{i\vec{k}\cdot\vec{x}} g_{\vec{k}}(z) \to z^{d-\Delta} \int d^d k e^{i\vec{k}\cdot\vec{x}} = z^{d-\Delta}\tilde{\phi}(\vec{x}) \qquad (z \to 0) \tag{5}$$

Thus, this is the correct solution with the given boundary condition :

$$\phi(\vec{x}, z) = \int d^d k \tilde{\phi}'(\vec{k}) e^{i\vec{k}\cdot\vec{x}} g_{\vec{k}}(z) \tag{6}$$

(c) It is easiest to show (equivalently) that

$$g_{\vec{k}}(z) \propto \int d^d y e^{-i\vec{k}\cdot\vec{y}} \left(\frac{z}{z^2 + \vec{y}^2}\right)^\Delta \tag{7}$$

Let us prove it by computing the right-hand-side. Using the integral definition of the gamma function, we write

$$\frac{1}{(z^2+\vec{y}^2)^\Delta} = \frac{1}{\Gamma(\Delta)} \int_0^\infty d\alpha \alpha^{\Delta-1} e^{-\alpha(z^2+\vec{y}^2)} \tag{8}$$

This allows to swap the $\vec{y}$ and α integrals and complete the square :

$$\begin{aligned} \int d^d y e^{-i\vec{k}\cdot\vec{y}} \left(\frac{z}{z^2+\vec{y}^2}\right)^\Delta &= \frac{z^\Delta}{\Gamma(\Delta)} \int_0^\infty d\alpha e^{-\alpha z^2} \alpha^{\Delta-1} \int d^d x e^{-\alpha(\vec{x}-\frac{i}{2\alpha}\vec{k})^2 - \frac{1}{4\alpha}\vec{k}^2} \\ &= \frac{z^\Delta}{\Gamma(\Delta)} \pi^{d/2} \int_0^\infty d\alpha e^{-\alpha z^2} \alpha^{\Delta-d/2-1} e^{-\vec{k}^2/(4\alpha)} \end{aligned} \tag{9}$$

Rescaling $\alpha \to \alpha/z^2$ gives

$$\int d^d y e^{-i\vec{k}\cdot\vec{y}} \left(\frac{z}{z^2+\vec{y}^2}\right)^\Delta = \frac{z^\Delta}{\Gamma(\Delta)} \pi^{d/2} z^{-2\Delta+d} \int_0^\infty d\alpha \alpha^{\Delta-d/2-1} e^{-\alpha - \frac{\vec{k}^2 z^2}{4\alpha}} \tag{10}$$

It turns out that this last integral is an integral representation of the modified Bessel function K. More specifically,

$$\int_0^\infty d\alpha \alpha^{\lambda-1} e^{-\alpha-\frac{x^2}{4\alpha}} = 2^{-\lambda+1} |x|^\lambda K_\lambda(|x|) \tag{11}$$

Using this property, and after bringing all factors together, this gives

$$\int d^d y e^{-i\vec{k}\cdot\vec{y}} \left(\frac{z}{z^2+\vec{y}^2}\right)^\Delta = \pi^{d/2} \frac{\Gamma(\Delta-\frac{d}{2})}{\Gamma(\Delta)} g_{\vec{k}}(z) \tag{12}$$

(d) Starting from

$$\phi(\vec{x}, z) = \int d^d k \tilde{\phi}'(\vec{k}) e^{i\vec{k}\cdot\vec{x}} g_{\vec{k}}(z) \tag{13}$$

We insert our previously derived result

$$g_{\vec{k}}(z) = \frac{1}{\pi^{d/2}} \frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)} \int d^d y e^{-i\vec{k}\cdot\vec{y}} \left(\frac{z}{z^2+\vec{y}^2}\right)^\Delta \tag{14}$$

obtaining

$$\phi(\vec{x}, z) = \frac{1}{\pi^{d/2}} \frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)} \int d^d y d^d k e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \tilde{\phi}'(\vec{k}) \left(\frac{z}{z^2+\vec{y}^2}\right)^\Delta \tag{15}$$

which doing some shifts and sign reversals reads

$$\phi(\vec{x}, z) = \frac{1}{\pi^{d/2}} \frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)} \int d^d y d^d k e^{i\vec{k}\cdot\vec{y}} \tilde{\phi}'(\vec{k}) \left(\frac{z}{z^2+(\vec{x}-\vec{y})^2}\right)^\Delta \tag{16}$$

We recognize the inverse Fourier transform which gives back $\tilde{\phi}$. Thus,

$$\phi(\vec{x}, z) = \int d^d y K(\vec{x}, z; \vec{y}) \tilde{\phi}(y) \tag{17}$$

where

$$K_\Delta(\vec{x}, z; \vec{y}) \equiv \frac{1}{\pi^{d/2}} \frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)} \left(\frac{z}{z^2+(\vec{x}-\vec{y})^2}\right)^\Delta . \tag{18}$$

2. Correlation functions

(a) Starting from

$$S \sim \frac{1}{G} \int d^{10}x \sqrt{-g}(R + c_1 \alpha' R^2 + c_2 \alpha' R_{\mu\nu} R^{\mu\nu} + \dots) \tag{19}$$

we rescale the metric by making the AdS scale R_{Ads} appear, $g_{\mu\nu} = R_{\text{AdS}}^2 \tilde{g}_{\mu\nu}$. This gives

$$R = R_{\text{AdS}}^{-2} \tilde{R} \qquad R_{\mu\nu} R^{\mu\nu} = R_{\text{AdS}}^{-4} \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} \qquad \sqrt{-g} = R_{\text{AdS}}^{10} \sqrt{-\tilde{g}} \tag{20}$$

Altogether,

$$S \sim \frac{R_{\text{AdS}}^8}{G} \int d^{10}x \sqrt{-\tilde{g}}(\tilde{R} + c_1 \alpha' R_{\text{AdS}}^{-2} \tilde{R}^2 + c_2 \alpha' R_{\text{AdS}}^{-2} \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} + \dots) \tag{21}$$

Let us call this prefactor

$$\frac{1}{\tilde{G}} \equiv \frac{R_{\text{AdS}}^8}{G} = \frac{R_{\text{AdS}}^8}{\ell_p^8} \sim N^2 \tag{22}$$

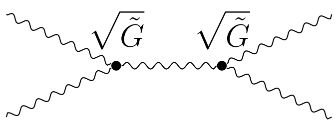
To compute interactions on top of $\tilde{g}_{\mu\nu}^{(\text{AdS})}$, we expand

$$\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}^{(\text{AdS})} + \sqrt{\tilde{G}} h_{\mu\nu} \tag{23}$$

This gives typically

$$S \sim \int d^{10}x \; \underbrace{(\partial h)^2 + \sqrt{\tilde{G}} h \partial h \partial h + \dots}_{\text{Einstein-Hilbert}} + \underbrace{\alpha' R_{\text{AdS}}^{-2} (\partial^2 h \partial^2 h + \sqrt{\tilde{G}} h \partial^2 h \partial^2 h + \dots) + \dots}_{\text{Higher curvature corrections}} \tag{24}$$

The corrections in $\sqrt{\tilde{G}}$ contribute to diagrams such as



$$\sim \tilde{G} \sim \frac{1}{N^2} \tag{25}$$

(and similarily for UV-divergent loop diagrams). Thus, $1/N^2$ corrections correspond to the regime where we need a UV completion to Einstein-Hilbert gravity to be able to make predictions from loop diagrams. These corrections correspond to *quantum gravity* corrections.

On the other hand, $\alpha' R_{\text{AdS}}^{-2}$ controls higher curvature corrections, where the Einstein-Hilbert action itself is not sufficient. In terms of string theory parameters,

$$\alpha' R_{\text{AdS}}^{-2} \sim \ell_s^2 (g_s \ell_s^4 N)^{-1/2} \sim (g_s N)^{-1/2} \sim \lambda^{-1/2} \tag{26}$$

Thus, Einstein-Hilbert is sufficient when $\lambda \to \infty$, and $1/\lambda$ corrections correspond to higher curvature, *stringy*, corrections.

(b) We will write formulas in generic d, where $d=4$ is the case of interest. We want to compute

$$\langle \mathcal{O}_1(\vec{x}_1)\mathcal{O}_2(\vec{x}_2)\mathcal{O}_3(\vec{x}_3)\rangle = -\frac{\lambda}{\pi^6}\left(\prod_i \frac{\Gamma(\Delta_i)}{\Gamma(\Delta_i-2)}\right) I \tag{27}$$

where the integral I is

$$\begin{aligned} I &= \int dz d^d x \sqrt{g}\frac{z^{\Delta_1}z^{\Delta_2}z^{\Delta_3}}{(z^2+(\vec{x}-\vec{x}_1)^2)^{\Delta_1}(z^2+(\vec{x}-\vec{x}_2)^2)^{\Delta_2}(z^2+(\vec{x}-\vec{x}_3)^2)^{\Delta_3}} \\ &= \int dz d^d x \frac{1}{z^{d+1}}\frac{1}{(z+(\vec{x}-\vec{x}_1)^2/z)^{\Delta_1}(z+(\vec{x}-\vec{x}_2)^2/z)^{\Delta_2}(z+(\vec{x}-\vec{x}_3)^2/z)^{\Delta_3}} \end{aligned} \tag{28}$$

Let us write the integrand as

$$\int d^d x \prod_{i=1}^{3}\frac{1}{(z+(\vec{x}-\vec{x}_i)^2/z)^{\Delta_i}} = \prod_i \frac{1}{\Gamma(\Delta_i)}\int_0^\infty ds_1 ds_2 ds_3 \prod_i s_i^{\Delta_i-1}e^{-\sum_i s_i(z+(\vec{x}-\vec{x}_i)^2/z)} \tag{29}$$

We then complete the square in $\vec{x}$, namely

$$\begin{aligned} \sum_i s_i(z+(\vec{x}-\vec{x}_i)^2/z) = \frac{1}{z}(\sum_i s_i)\left(\vec{x}-\frac{\sum_j s_j\vec{x}_j}{\sum_i s_i}\right)^2 \\ -\frac{(\sum_j s_j\vec{x}_j)^2}{z\sum_i s_i}+\frac{1}{z}\sum_i s_i x_i^2 \end{aligned} \tag{30}$$

This allows to compute the integral over $\vec{x}$, yielding

$$I = \prod_i \frac{1}{\Gamma(\Delta_i)}\int dz ds_i \prod_i s_i^{\Delta_i-1}\frac{1}{z^{d+1}}\frac{\pi^{d/2}z^{d/2}}{(\sum_i s_i)^{d/2}}\exp\left(-\sum_i \lambda_i z-\sum_i \frac{s_i\vec{x}_i^2}{z}+\frac{(\sum_j s_j\vec{x}_j)^2}{\sum_i s_i z}\right) \tag{31}$$

Rescaling $z \to z/\sum_i s_i$, this gives

$$I = \pi^{d/2}\prod_i \frac{1}{\Gamma(\Delta_i)}\int_0^\infty dz ds_i \frac{1}{z^{d/2+1}}\exp\left(-z-\frac{s_1s_2\vec{x}_{12}^2+s_1s_3\vec{x}_{13}^2+s_2s_3\vec{x}_{23}^2}{z}\right) \tag{32}$$

where $\vec{x}_{ij}\equiv \vec{x}_i-\vec{x}_j$. This integral can be expressed in terms of Γ functions. To see it, we do the change of variables from (z,s_1,s_2,s_3) to (z,t_1,t_2,t_3) defined by

$$s_i = \frac{\sqrt{zt_1t_2t_3}}{t_i} \tag{33}$$

The Jacobian can be computed straight-forwardly and reads

$$\left|\frac{\partial(z,s_i)}{\partial(z,t_i)}\right| = \frac{z^{3/2}}{2\sqrt{t_1t_2t_3}} \tag{34}$$

The nice property of this change of variables is that

$$\frac{s_1s_2\vec{x}_{12}^2+s_1s_3\vec{x}_{13}^2+s_2s_3\vec{x}_{23}^2}{z} = t_3\vec{x}_{12}^2+t_2\vec{x}_{13}^2+t_1\vec{x}_{23}^2 \tag{35}$$

Altogether,

$$I = \frac{\pi^{d/2}}{2}\prod_i \frac{1}{\Gamma(\Delta_i)}\int_0^\infty dz dt_i z^{\frac{\Delta_1+\Delta_2+\Delta_3-d}{2}-1} t_1^{-1+\frac{\Delta_2+\Delta_3-\Delta_1}{2}} t_2^{-1+\frac{\Delta_1+\Delta_3-\Delta_2}{2}} \\ t_3^{-1+\frac{\Delta_1+\Delta_2-\Delta_3}{2}} e^{-z-t_3\vec{x}_{12}^2-t_2\vec{x}_{13}^2-t_1\vec{x}_{23}^2} \tag{36}$$

Once rescaling $t_3 \to t_3/\vec{x}_{12}^2$, and similarly for t_1 and t_2, we recognize the definition of gamma functions. Thus, plugging back $d = 4$ (AdS_5) we obtain

$$\begin{aligned}\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle &= -\frac{\lambda}{\pi^6}\left(\prod_i \frac{\Gamma(\Delta_i)}{\Gamma(\Delta_i-2)}\right) I \\ &= \frac{\lambda a_1}{|\vec{x}_1-\vec{x}_2|^{\Delta_1+\Delta_2-\Delta_3}|\vec{x}_1-\vec{x}_3|^{\Delta_1+\Delta_3-\Delta_2}|\vec{x}_2-\vec{x}_3|^{\Delta_2+\Delta_3-\Delta_1}}\end{aligned} \tag{37}$$

with

$$a_1 = -\frac{\Gamma\left[\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)\right]\Gamma\left[\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2)\right]\Gamma\left[\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)\right]}{2\pi^4\Gamma(\Delta_1-2)\Gamma(\Delta_2-2)\Gamma(\Delta_3-2)} \\ \cdot\Gamma\left[\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3)-2\right] \tag{38}$$

3. Two-point function of a massive scalar on non-rotating BTZ

It is convenient to use the coordinate $r = r_+\cosh\mu$, for which $r^2 - r_+^2 = r_+^2\sinh^2\mu$ and

$$ds^2 = -r_+^2\sinh^2\mu\, dt^2 + d\mu^2 + r_+^2\cosh^2\mu\, d\phi^2, \qquad \mu\in[0,\infty), \quad (\mu = 0 \text{ horizon}),\ (\mu\to\infty \text{ boundary}). \tag{39}$$

Define the dimensionless radial variable

$$z \equiv \tanh^2\mu \in [0,1), \qquad z = 0 \text{ (horizon)}, \qquad z \to 1 \text{ (boundary)}. \tag{40}$$

(a) Radial equation and hypergeometric form. With the mode ansatz

$$\Phi = e^{-i\omega t + ik\phi}R(\mu)$$

the KG equation $(\Box - m^2)\Phi = 0$ reduces to an ODE for R. In terms of $z = \tanh^2\mu$ this ODE can be brought to the hypergeometric form

$$z(1-z)R''(z) + (1-z)R'(z) + \left[\frac{k_+^2}{4z} - \frac{k_-^2}{4} - \frac{m^2}{4(1-z)}\right]R(z) = 0, \tag{41}$$

where it is useful to package the Fourier labels into

$$k_+ \equiv \frac{\omega}{r_+}, \qquad k_- \equiv -\frac{k}{r_+}, \qquad \Rightarrow \qquad k_+ \pm k_- = \frac{\omega \mp k}{r_+}. \tag{42}$$

(Equivalently, one may work directly in (t, r, ϕ) and use $z = 1 - r_+^2/r^2$; the resulting equation is the same hypergeometric equation.)

(b) Ingoing solution and near-boundary coefficients. Our ODE takes the form

$$\begin{aligned} &z(1-z)\,R''(z) + \Big[(c-2\alpha) + \big(2(\alpha+\beta) - a - b - 1\big)z\Big]\,R'(z) \\ &+ \Big[\frac{\alpha(\alpha+1-c)}{z} + \frac{\beta(\beta+c-a-b)}{1-z} + (\alpha+\beta)(a+b-\alpha-\beta) - ab\Big]\,R(z) = 0. \end{aligned} \tag{43}$$

with parameters

$$\alpha = -\frac{ik_+}{2} = -\frac{i\omega}{2r_+}, \qquad \beta = \frac{1}{2}\Big(1 - \sqrt{1+m^2}\Big), \tag{44}$$

$$a = \beta + \frac{k_+ - k_-}{2i} = \beta - \frac{i(\omega+k)}{2r_+}, \qquad b = \beta + \frac{k_+ + k_-}{2i} = \beta - \frac{i(\omega-k)}{2r_+}, \qquad c = 1 + 2\alpha = 1 - \frac{i\omega}{r_+}. \tag{45}$$

Two linearly independent solutions are

$$R_1(z) = z^{\alpha}(1-z)^{\beta}\,{}_2F_1(a,b;c;z), \tag{46}$$

and

$$R_2(z) = z^{\alpha+1-c}(1-z)^{\beta}\,{}_2F_1(a-c+1, b-c+1; 2-c; z), \tag{47}$$

At the horizon $z \approx 0$, we have that

$$R_1 \sim z^{\alpha} = z^{-i\omega/(2r_+)}, \qquad R_2 \sim z^{-\alpha} = z^{i\omega/(2r_+)} \tag{48}$$

Thus, R_1 is the in-going mode since $e^{-i\omega t} z^{-i\omega/(2r_+)}$ remains of constant phase when t increases and z decreases (goes inside the horizon), while R_2 is the outgoing mode. We thus choose $R = R_1$.

Near the boundary $z \to 1$, use the standard hypergeometric transformation formula to expand

$$\begin{aligned} {}_2F_1(a,b;c;z) = &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b;a+b-c+1;1-z) \\ &+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,{}_2F_1(c-a,c-b;c-a-b+1;1-z). \end{aligned} \tag{49}$$

Since ${}_2F_1(\cdots;1-z) \to 1$ as $z \to 1$, this gives the leading behavior

$$R(z) \underset{z\to 1}{\sim} A(\omega,k)\,(1-z)^{\beta} + B(\omega,k)\,(1-z)^{1-\beta}, \tag{50}$$

with

$$A(\omega,k) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad B(\omega,k) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \tag{51}$$

For $m^2 > 0$, one has $\beta < 0$, hence the $(1-z)^{\beta}$ term diverges at the boundary and is the non-normalisable/source piece, while $(1-z)^{1-\beta}$ is the normalisable/response piece.

It is useful to introduce the CFT weight

$$\Delta = 1 + \sqrt{1+m^2}, \qquad h = \bar{h} = \frac{\Delta}{2} = \frac{1+\sqrt{1+m^2}}{2} \qquad \Rightarrow \qquad 1-\beta = h, \quad \beta = 1-h. \tag{52}$$

(c) Retarded Green's function and quasi-normal poles. With ingoing boundary conditions at the horizon, the real-time AdS/CFT prescription gives (up to an overall, ω-independent normalization)

$$G_R(\omega, k) \;\propto\; \frac{B(\omega, k)}{A(\omega, k)}. \tag{53}$$

Using $\beta = 1 - h$ and writing everything in terms of the temperature $T = \frac{r_+}{2\pi}$ (so $2r_+ = 4\pi T$), one finds the compact form

$$G_R(\omega, k) \;\propto\; \frac{\Gamma\big(h - i\frac{\omega-k}{4\pi T}\big)\;\Gamma\big(h - i\frac{\omega+k}{4\pi T}\big)}{\Gamma\big(1 - h - i\frac{\omega-k}{4\pi T}\big)\;\Gamma\big(1 - h - i\frac{\omega+k}{4\pi T}\big)}. \tag{54}$$

Poles of G_R occur when the source coefficient vanishes, i.e. when $A(\omega, k) = 0$. Since $A \propto 1/(\Gamma(c-a)\Gamma(c-b))$, and $1/\Gamma(x)$ has zeros at $x = 0, -1, -2, \ldots$, the poles are at

$$c - a = -n \quad \text{or} \quad c - b = -n, \qquad n = 0, 1, 2, \ldots \tag{55}$$

Using

$$c - a = h - i\frac{\omega - k}{2r_+} = h - i\frac{\omega - k}{4\pi T}, \qquad c - b = h - i\frac{\omega + k}{2r_+} = h - i\frac{\omega + k}{4\pi T}, \tag{56}$$

we obtain the two towers of quasi-normal/pole frequencies (in the lower half-plane)

$$\omega_n^{(\pm)} = \pm k - 4\pi i\, T\,(n + h), \qquad n = 0, 1, 2, \ldots \tag{57}$$

which have $\mathrm{Im}(\omega) < 0$ and hence describe decay/relaxation.

(d) Match to the thermal CFT correlator.

Thermal expectation values and KMS. Let us first recall or introduce a few concepts. We define thermal expectation values in the Hilbert space by

$$\langle \mathcal{A} \rangle_T \;\equiv\; \frac{1}{Z(\beta)}\,\mathrm{Tr}\big(e^{-\beta H}\,\mathcal{A}\big)\,, \qquad Z(\beta) = \mathrm{Tr}\big(e^{-\beta H}\big), \qquad \beta = \frac{1}{T}. \tag{58}$$

For two-point functions we distinguish the Wightman correlators

$$G^{>}(t, \sigma) \equiv \langle \mathcal{O}(t, \sigma)\mathcal{O}(0, 0)\rangle_T, \qquad G^{<}(t, \sigma) \equiv \langle \mathcal{O}(0, 0)\mathcal{O}(t, \sigma)\rangle_T, \tag{59}$$

which are *not* equal in general since the operators do not commute. Using cyclicity of the trace and $\mathcal{O}(t) = e^{iHt}\mathcal{O}\,e^{-iHt}$, one can instead derive the KMS condition:

$$\begin{aligned} G^{>}(t, \sigma) &= \frac{1}{Z}\,\mathrm{Tr}\Big(e^{-\beta H} e^{iHt}\mathcal{O}(\sigma)e^{-iHt}\mathcal{O}(0)\Big) \\ &= \frac{1}{Z}\,\mathrm{Tr}\Big(\mathcal{O}(0)\,e^{-\beta H} e^{iHt}\mathcal{O}(\sigma)e^{-iHt}\Big) \\ &= \frac{1}{Z}\,\mathrm{Tr}\Big(e^{-\beta H}\mathcal{O}(0)\,e^{iH(t-i\beta)}\mathcal{O}(\sigma)e^{-iH(t-i\beta)}\Big) \;=\; G^{<}(t - i\beta, \sigma), \end{aligned} \tag{60}$$

i.e.

$$G^{>}(t, \sigma) = G^{<}(t - i\beta, \sigma)\;. \tag{61}$$

Euclidean correlator The Euclidean thermal correlator computed by the path integral on the cylinder is the τ-*ordered* correlator

$$G_E(\tau,\sigma) \equiv \langle \mathcal{T}_\tau\, \mathcal{O}_E(\tau,\sigma)\mathcal{O}_E(0,0)\rangle_T, \qquad \tau\sim\tau+\beta. \tag{62}$$

On the complex plane, conformal invariance fixes

$$\langle \mathcal{O}(z,\bar z)\mathcal{O}(0)\rangle \propto \frac{1}{z^{2h}\bar z^{2\bar h}}. \tag{63}$$

Mapping the plane to the thermal cylinder (Euclidean time periodic with period $1/T$) via

$$z = e^{2\pi T w}, \qquad w=\sigma+i\tau, \tag{64}$$

gives (up to normalization)

$$G_E(\tau,\sigma) \;\propto\; \left(\frac{\pi T}{\sin\big(\pi T(\tau+i\sigma)\big)}\right)^{2h}\left(\frac{\pi T}{\sin\big(\pi T(\tau-i\sigma)\big)}\right)^{2\bar h}. \tag{65}$$

Retarded correlator and how it is obtained. The retarded correlator is defined by linear response:

$$G_R(t,\sigma) \equiv -i\,\theta(t)\,\langle[\mathcal{O}(t,\sigma),\mathcal{O}(0,0)]\rangle_T = -i\,\theta(t)\,\big(G^{>}(t,\sigma)-G^{<}(t,\sigma)\big). \tag{66}$$

In particular, $G_R(t,\sigma)=0$ for $t<0$, and therefore $G_R(\omega,k)$ is analytic for $\operatorname{Im}\omega>0$ (which forces poles to lie in the lower-half ω-plane for a stable thermal state). Note that G_R in *position space* is **not** a simple analytic continuation of G_E. It has causal support $G_R(t<0,\sigma)=0$. In contrast, the Euclidean correlator $G_E(\tau,\sigma)$ is a thermal (τ-ordered) quantity on the circle and its naive continuation $\tau\to it+\epsilon$ gives (a branch of) a Wightman function, not the commutator with $\theta(t)$. Therefore G_R is *not* obtained by a direct pointwise analytic continuation of $G_E(\tau,\sigma)$ in position space. The correct relation is formulated in momentum space.
To see it, Fourier transform in σ:

$$\mathcal{O}_k(t) \equiv \int_{-\infty}^{\infty} d\sigma\, e^{-ik\sigma}\,\mathcal{O}(t,\sigma), \qquad G_{E/R}(\cdot,k) \equiv \int d\sigma\, e^{-ik\sigma}\, G_{E/R}(\cdot,\sigma). \tag{67}$$

For $0<\tau<\beta$ insert a complete set of energy eigenstates $H|a\rangle = E_a|a\rangle$:

$$\begin{aligned}G_E(\tau,k) &= \frac{1}{Z}\operatorname{Tr}\big(e^{-\beta H}\,\mathcal{O}_k(\tau)\mathcal{O}_{-k}(0)\big) = \frac{1}{Z}\sum_{a,b} e^{-\beta E_a}\, e^{(E_a-E_b)\tau}\,\langle a|\mathcal{O}_k|b\rangle\langle b|\mathcal{O}_{-k}|a\rangle \\ &= \frac{1}{Z}\sum_{a,b} e^{-\beta E_a}\, e^{-(E_b-E_a)\tau}\,|\mathcal{O}_{ab}(k)|^2, \qquad |\mathcal{O}_{ab}(k)|^2 \equiv \langle a|\mathcal{O}_k|b\rangle\langle b|\mathcal{O}_{-k}|a\rangle.\end{aligned} \tag{68}$$

Define bosonic Matsubara frequencies $\omega_n = 2\pi n T$ and the Matsubara transform

$$G_E(i\omega_n,k) \equiv \int_0^\beta d\tau\, e^{i\omega_n\tau}\, G_E(\tau,k). \tag{69}$$

Perform the τ integral term-by-term:

$$\begin{aligned}\int_0^\beta d\tau\, e^{(i\omega_n-(E_b-E_a))\tau} &= \frac{e^{(i\omega_n-(E_b-E_a))\beta}-1}{i\omega_n-(E_b-E_a)} = \frac{e^{-\beta(E_b-E_a)}-1}{i\omega_n-(E_b-E_a)} \quad (\text{since } e^{i\omega_n\beta}=1) \\ &= \frac{e^{-\beta E_b}-e^{-\beta E_a}}{i\omega_n-(E_b-E_a)}\, e^{+\beta E_a}.\end{aligned} \tag{70}$$

Thus

$$G_E(i\omega_n,k)=\frac{1}{Z}\sum_{a,b}\frac{e^{-\beta E_a}-e^{-\beta E_b}}{i\omega_n-(E_b-E_a)}\,|\mathcal{O}_{ab}(k)|^2. \tag{71}$$

Now let us look at the retarded correlator. Start from (66) and insert the same basis:

$$\langle[\mathcal{O}_k(t),\mathcal{O}_{-k}(0)]\rangle_T=\frac{1}{Z}\sum_{a,b}\left(e^{-\beta E_a}-e^{-\beta E_b}\right)e^{-i(E_b-E_a)t}\,|\mathcal{O}_{ab}(k)|^2. \tag{72}$$

Fourier transforming (66) in time gives

$$\begin{aligned}G_R(\omega,k)&=\int_{-\infty}^{\infty}dt\,e^{i\omega t}G_R(t,k)=-i\int_0^{\infty}dt\,e^{i\omega t}\,\langle[\mathcal{O}_k(t),\mathcal{O}_{-k}(0)]\rangle_T\\&=\frac{1}{Z}\sum_{a,b}\left(e^{-\beta E_a}-e^{-\beta E_b}\right)|\mathcal{O}_{ab}(k)|^2\left[-i\int_0^{\infty}dt\,e^{i(\omega-(E_b-E_a))t}\right].\end{aligned} \tag{73}$$

Using the standard distributional identity

$$\int_0^{\infty}dt\,e^{i(\omega-\Delta E)t}=\frac{i}{\omega-\Delta E+i0^+}, \tag{74}$$

we obtain

$$G_R(\omega,k)=\frac{1}{Z}\sum_{a,b}\frac{e^{-\beta E_a}-e^{-\beta E_b}}{\omega-(E_b-E_a)+i0^+}\,|\mathcal{O}_{ab}(k)|^2. \tag{75}$$

Comparing (71) and (75) shows that they are the *same* rational function of the complex frequency variable, evaluated at different arguments, with the retarded boundary condition encoded by $+i0^+$:

$$G_R(\omega,k)=G_E(i\omega_n,k)\Big|_{i\omega_n\to\omega+i0^+}. \tag{76}$$

Explicit Matsubara transform of the CFT Euclidean correlator
We now compute $G_E(i\omega_n,k)$ directly from (65) for the scalar case $h=\bar{h}$. First use

$$\sin(a+ib)\sin(a-ib)=\frac{\cosh(2b)-\cos(2a)}{2}. \tag{77}$$

With $a=\pi T\tau$ and $b=\pi T\sigma$, (65) becomes (up to an overall constant)

$$G_E(\tau,\sigma)\;\propto\;(\pi T)^{4h}\,2^{2h}\,[\cosh(2\pi T\sigma)-\cos(2\pi T\tau)]^{-2h}. \tag{78}$$

Introduce dimensionless variables

$$\theta\equiv 2\pi T\tau\in[0,2\pi),\qquad x\equiv 2\pi T\sigma\in\mathbb{R},\qquad \kappa\equiv\frac{k}{2\pi T},\qquad \omega_n=2\pi nT. \tag{79}$$

Then the Matsubara transform (including the Jacobian $d\tau\,d\sigma=\frac{d\theta\,dx}{(2\pi T)^2}$) reads

$$G_E(i\omega_n,k)\;\propto\;(2\pi T)^{4h-2}\int_{-\infty}^{\infty}dx\,e^{-i\kappa x}\int_0^{2\pi}d\theta\,e^{in\theta}\,[\cosh x-\cos\theta]^{-2h}. \tag{80}$$

Use, for $\mathrm{Re}\,h>0$,

$$[\cosh x-\cos\theta]^{-2h}=\frac{1}{\Gamma(2h)}\int_0^{\infty}ds\;s^{2h-1}\,e^{-s(\cosh x-\cos\theta)}=\frac{1}{\Gamma(2h)}\int_0^{\infty}ds\;s^{2h-1}\,e^{-s\cosh x}\,e^{s\cos\theta}. \tag{81}$$

Insert (81) into (80) and exchange integrals:

$$G_E(i\omega_n,k) \;\propto\; (2\pi T)^{4h-2}\frac{1}{\Gamma(2h)}\int_0^\infty ds\; s^{2h-1}\left[\int_0^{2\pi} d\theta\, e^{in\theta}e^{s\cos\theta}\right]\left[\int_{-\infty}^{\infty} dx\, e^{-i\kappa x}e^{-s\cosh x}\right]. \tag{82}$$

The generating function of modified Bessel functions gives, for integer n,

$$\int_0^{2\pi} d\theta\, e^{in\theta}e^{s\cos\theta} = 2\pi\, I_n(s). \tag{83}$$

Use the standard representation

$$K_{i\kappa}(s) = \frac{1}{2}\int_{-\infty}^{\infty} dx\, e^{-s\cosh x}\, e^{i\kappa x} \qquad (\kappa\in\mathbb{R},\; s>0), \tag{84}$$

so

$$\int_{-\infty}^{\infty} dx\, e^{-i\kappa x}e^{-s\cosh x} = 2\, K_{i\kappa}(s). \tag{85}$$

Putting these together, we reduce the Matsubara transform to a single integral:

$$G_E(i\omega_n,k) \;\propto\; (2\pi T)^{4h-2}\frac{4\pi}{\Gamma(2h)}\int_0^\infty ds\; s^{2h-1}\, I_n(s)\, K_{i\kappa}(s), \qquad (n\geq 0). \tag{86}$$

The remaining integral is a standard special-function identity (valid in a convergence strip, then by analytic continuation in h):

$$\int_0^\infty ds\; s^{2h-1} I_n(s) K_{i\kappa}(s) = 2^{2h-2}\Gamma(1-2h)\,\frac{\Gamma\big(h+\frac{n}{2}+\frac{i\kappa}{2}\big)\,\Gamma\big(h+\frac{n}{2}-\frac{i\kappa}{2}\big)}{\Gamma\big(1-h+\frac{n}{2}+\frac{i\kappa}{2}\big)\,\Gamma\big(1-h+\frac{n}{2}-\frac{i\kappa}{2}\big)}, \tag{87}$$

so, absorbing overall h-dependent constants into normalization,

$$G_E(i\omega_n,k) \;\propto\; \frac{\Gamma\big(h+\frac{n}{2}+\frac{i\kappa}{2}\big)\,\Gamma\big(h+\frac{n}{2}-\frac{i\kappa}{2}\big)}{\Gamma\big(1-h+\frac{n}{2}+\frac{i\kappa}{2}\big)\,\Gamma\big(1-h+\frac{n}{2}-\frac{i\kappa}{2}\big)}, \qquad n\geq 0. \tag{88}$$

Since $\kappa=\frac{k}{2\pi T}$ and $\frac{n}{2}=\frac{\omega_n}{4\pi T}$, this is equivalently

$$G_E(i\omega_n,k) \;\propto\; \frac{\Gamma\big(h+\frac{\omega_n+ik}{4\pi T}\big)\,\Gamma\big(h+\frac{\omega_n-ik}{4\pi T}\big)}{\Gamma\big(1-h+\frac{\omega_n+ik}{4\pi T}\big)\,\Gamma\big(1-h+\frac{\omega_n-ik}{4\pi T}\big)}. \tag{89}$$

Wick rotation to the retarded correlator and pole structure. Using (76) and continuing from $n>0$,

$$G_R(\omega,k) = G_E(i\omega_n,k)\Big|_{i\omega_n\to\omega+i0^+} \qquad\Longleftrightarrow\qquad \omega_n\to -i(\omega+i0^+). \tag{90}$$

Applying this to (89) gives, up to an overall factor,

$$G_R(\omega,k) \;\propto\; \frac{\Gamma\big(h-i\frac{\omega-k}{4\pi T}\big)\,\Gamma\big(h-i\frac{\omega+k}{4\pi T}\big)}{\Gamma\big(1-h-i\frac{\omega-k}{4\pi T}\big)\,\Gamma\big(1-h-i\frac{\omega+k}{4\pi T}\big)}. \tag{91}$$

Poles come from the numerator Gamma functions at non-positive integers:

$$h-i\frac{\omega\mp k}{4\pi T} = -m, \qquad m=0,1,2,\ldots, \tag{92}$$

hence the two towers

$$\omega = +k-4\pi iT(m+h), \qquad \omega=-k-4\pi iT(m+h), \qquad m=0,1,2,\ldots \tag{93}$$

(for a scalar $h=\bar{h}$), in exact agreement with the bulk quasi-normal/pole spectrum found in part (c).